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THE EXTENDED k-MITTAG-LEFFLER FUNCTION AND ITS PROPERTIES

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ABSTRACT. In this present paper, our aim is to derive the extended k-Mittag-Leffler function by using the extended k-beta function (Mubeen *et al.* in J. math. anal. Volume 7 Issue 5(2016), 118-131.) and define some integral representation this newly defined function. Also, we introduce the extended k-fractional derivative formula and show that the extended k-fractional derivative k-fractional of the k-Mittag-Leffler gives the extended k-Mittag-Leffler function.

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1. INTRODUCTION

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ which can be defined as the following form:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, z \in \mathbb{C}; \Re(\alpha) > 0 \tag{1}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z, \beta \in \mathbb{C}; \Re(\alpha) > 0, \tag{2}$$

respectively. For further details of Mittag-Leffler function such as generalizations and applications , the readers may refer to the work of researchers (for example) the Džrbašjan [2], Kilbas and Saigo [10], Gorenflo and Mainardi [5] , Gorenflo et al. ([6, 8]), Kilbas et al. [11] and Saigo and Kilbas [16]. In recent years, the Mittag-Leffler function (1) and the various generalizations of this function have been numerically investigated in the complex plane (see [9, 17]). A generalization of this Mittag-Leffler function $E_{\alpha,\beta}(z)$ of (2) was also introduced by the researcher Prabhakar [15] as follows:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, z, \beta \in \mathbb{C}; \Re(\alpha) > 0, \tag{3}$$

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where $(\gamma)_n$ denote the well known Pochhammer's symbol which is defined by:

$$(\gamma)_n = \begin{cases} 1, (n = 0, \gamma \in \mathbb{C}) \\ \gamma(\gamma+1)\cdots(\gamma+n-1), (n \in \mathbb{N}, \gamma \in \mathbb{C}). \end{cases}$$

Obviously, the following special cases are satisfied:

$$E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = E^1_{\alpha,1}(z) = E_\alpha(z). \tag{4}$$

Recently many researchers have investigated the importance and great consideration of Mittag-Leffler function in the theory of special functions for exploring the generalization and some applications. Many extensions for these functions are found in [7],[18]-[21]. Srivastava and Tomovski [22] have defined further generalization of the Mittag-Leffler function $E^\gamma_{\alpha,\beta}(z)$ of (3), which is defined as:

$$E^{\gamma,\kappa}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\kappa}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{5}$$

where $z, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha) > \max\{0, \Re(\kappa) - 1\}$; $\Re(\kappa) > 0$.
A useful generalization of the Mittag-Leffler function called k-Mittag-Leffler function $E^\gamma_{k,\alpha,\beta}(z)$ defined in [4] and its series representation is given by:

$$E^\gamma_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \tag{6}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $k > 0$ and $(\gamma)_{n,k}$ is the Pochhammer k-symbol defined as:

$$(\gamma)_{n,k} = \begin{cases} 1, (n = 0, \gamma \in \mathbb{C}) \\ \gamma(\gamma+k)\cdots(\gamma+(n-1)k), (n \in \mathbb{N}, \gamma \in \mathbb{C}, k > 0). \end{cases} \tag{7}$$

In this paper, we extend the k-Mittag-Leffler function $E^\gamma_{k,\alpha,\beta}(z)$ defined in (6) in the following way. Since

$$E^\gamma_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!},$$

using the fact that

$$\frac{(\gamma)_{n,k}}{(c)_{n,k}} = \frac{B_k(\gamma + nk, c - \gamma)}{B_k(\gamma, c - \gamma)},$$

we extend the k-Mittag-Leffler function as follows:

$$E^{\gamma;c}_{k,\alpha,\beta}(z;p) = \sum_{n=0}^{\infty} \frac{B_k(\gamma + nk, c - \gamma; p)}{B_k(\gamma, c - \gamma)} \frac{(c)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \tag{8}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $p \geq 0$, $k > 0$ and $B_k(x, y; p)$ is an extended k-beta function defined as:

$$B_k(x, y; p) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} \exp[-\frac{p^k}{kt(1-t)}]dt. \tag{9}$$

For further various properties of an extended k-beta function, the interesting readers may refer to see [12].

remark 1.1. *(i) Setting $p = 0$ in (8), then we will have the well known k-Mittag-Leffler function defined in (6).*
(ii) Setting $k = 1$ in (8), then we will have the well known an extended Mittag-Leffler function defined in [13].

The paper is organized as follows: In Section 2, we defined various integral representations of the newly defined extended k-Mittag-Leffler function. In Section 3, we obtain fractional integral and differential representation of an extended k-Mittag-Leffler function.

2. Some properties of the extended k-Mittag-Leffler function

In this section, we derive some basic properties of the extended k-Mittag-Leffler function. We begin with the following theorems which give integral representation, recurrence relations and Mellin transform.

Theorem 2.1. *For $k > 0$, the extended k-Mittag-Leffler function can be expressed in the following integral representation*

$$E_{k,\alpha,\beta}^{\gamma;c}(z;p) = \frac{1}{kB_k(\gamma, c-\gamma)} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1} e^{-\frac{p^k}{kt(1-t)}} E_{k,\alpha,\beta}^{c}(tz)dt, \tag{10}$$

where $p \geq 0$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$.

Proof. Using (9) in (8), we have

$$E_{k,\alpha,\beta}^{\gamma;c}(z;p) = \sum_{n=0}^{\infty} \left\{ \frac{1}{k} \int_0^1 t^{\frac{\gamma+nk}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1} e^{-\frac{p^k}{kt(1-t)}} dt \right\}$$
$$\times \frac{(c)_{n,k}}{B_k(\gamma, c-\gamma)} \frac{z^n}{\Gamma_k(\alpha n + \beta)}.$$

Interchanging the order of integration and summation in above equation, we have

$$E_{k,\alpha,\beta}^{\gamma;c}(z;p) = \frac{1}{k} \int_0^1 t^{\frac{\gamma+nk}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1} e^{-\frac{p^k}{kt(1-t)}}$$
$$\times \sum_{n=0}^{\infty} \frac{(c)_{n,k}}{B_k(\gamma, c-\gamma)} \frac{z^n}{\Gamma_k(\alpha n + \beta)} dt$$
$$= \frac{1}{k} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1} e^{-\frac{p^k}{kt(1-t)}}$$
$$\times \sum_{n=0}^{\infty} \frac{(c)_{n,k}}{B_k(\gamma, c-\gamma)} \frac{(tz)^n}{\Gamma_k(\alpha n + \beta)} dt$$
$$= \frac{1}{kB_k(\gamma, c-\gamma)} \int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1} e^{-\frac{p^k}{kt(1-t)}} E_{k,\alpha,\beta}^{c}(tz)dt,$$

which is the required proof. □

Theorem 2.2. *For $k > 0$, the following recurrence relation for the extended k-Mittag-Leffler function can be expressed in the following form*

$$E_{k,\alpha,\beta}^{\gamma;c}(z;p) = \beta E_{k,\alpha,\beta+k}^{\gamma;c}(z;p) + az\frac{d}{dz}E_{k,\alpha,\beta}^{\gamma;c}(z;p), \tag{11}$$

where $p \geq 0$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$.

Proof. It is easy to prove the above recurrence relation. □

Theorem 2.3. *The Mellin transform of the extended k-Mittag-Leffler function is expressed in the following form:*

$$\begin{aligned}\mathfrak{M}\left\{E_{k,\alpha,\beta}^{\gamma;c}(z;p);s\right\} &= \frac{\Gamma_k(s)\Gamma_k(c-\gamma+s)}{\Gamma_k(\gamma)\Gamma_k(c-\gamma)}\\ &\times {}_{k,2}\Psi_2\left[\begin{array}{c}(c,k),(\gamma+s,k);\\ \\ (\beta,\alpha),(c+2s,k)\end{array} ,z\right],\end{aligned} \tag{12}$$

where $p \geq 0$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and ${}_{k,2}\Psi_2$ the k-Fox-Wright function [1].

Proof. Taking the Mellin transform of the extended k-Mittag-Leffler function, we have

$$\mathfrak{M}\{\ E_{k,\alpha,\beta}^{\gamma;c}(z;p);s\ \} = \int_0^\infty p^{s-1}E_{k,\alpha,\beta}^{\gamma;c}(z;p)dp \tag{13}$$

Using (10) in (13), we obtain

$$\begin{aligned}\mathfrak{M}\left\{E_{k,\alpha,\beta}^{\gamma;c}(z;p);s\right\} &= \int_0^\infty p^{s-1}\left[\int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1}e^{-\frac{p^k}{kt(1-t)}}\right]\\ &\times E_{k,\alpha,\beta}^{c}(tz)dtdp.\end{aligned} \tag{14}$$

By interchanging the order of integration, we have

$$\begin{aligned}\mathfrak{M}\left\{E_{k,\alpha,\beta}^{\gamma;c}(z;p);s\right\} &= \frac{1}{kB_k(\gamma,c-\gamma)}\left[\int_0^1 t^{\frac{\gamma}{k}-1}(1-t)^{\frac{c-\gamma}{k}-1}E_{k,\alpha,\beta}^{c}(tz)\right]\\ &\times \int_0^\infty p^{s-1}e^{-\frac{p^k}{kt(1-t)}}dpdt.\end{aligned} \tag{15}$$

Consider the integral

$$\int_0^\infty p^{s-1}e^{-\frac{p^k}{kt(1-t)}}dp.$$

By using $u = \frac{p}{[t(1-t)]^{\frac{1}{k}}}$ in the above integral, we otained

$$\Gamma_k(s) = t^{\frac{s}{k}}(1-t)^{\frac{s}{k}}\int_0^\infty p^{s-1}e^{-\frac{u^k}{k}}du.$$

Using this result and the definition of k-Mittag-Leffler function in (15), we have

$$\mathfrak{M}\left\{ E_{k,\alpha,\beta}^{\gamma;c}(z;p);s \right\}$$

$$(16)\quad = \frac{\Gamma_k(s)}{kB_k(\gamma,c-\gamma)}\int_0^1 t^{\frac{\gamma+s}{k}-1}(1-t)^{\frac{c-\gamma+s}{k}-1}\sum_{n=0}^{\infty}\frac{(c)_{n,k}(tz)^n}{\Gamma_k(\alpha n+\beta)n!}dt.$$

Interchanging the order of integration and summation, we have

$$\mathfrak{M}\left\{ E_{k,\alpha,\beta}^{\gamma;c}(z;p);s \right\}$$

$$(17)\quad = \frac{\Gamma_k(s)}{B_k(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{(c)_{n,k}(z)^n}{\Gamma_k(\alpha n+\beta)n!}\frac{1}{k}\int_0^1 t^{\frac{\gamma+nk+s}{k}-1}(1-t)^{\frac{c-\gamma+s}{k}-1}dt$$

Using the definition of k-beta function, we have

$$\mathfrak{M}\left\{ E_{k,\alpha,\beta}^{\gamma;c}(z;p);s \right\}$$

$$(18)\quad = \frac{\Gamma_k(s)}{B_k(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{(c)_{n,k}(z)^n}{\Gamma_k(\alpha n+\beta)n!}\frac{\Gamma_k(\gamma+nk+s)\Gamma_k(c-\gamma+s)}{\Gamma_k(c+nk+2s)}.$$

This can be written as

$$\mathfrak{M}\left\{ E_{k,\alpha,\beta}^{\gamma;c}(z;p);s \right\}$$

$$= \frac{\Gamma_k(s)\Gamma_k(c-\gamma+s)}{\Gamma_k(c)B_k(\gamma,c-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma_k(c+nk)}{\Gamma_k(\alpha n+\beta)n!}\frac{\Gamma_k(\gamma+nk+s)}{\Gamma_k(c+nk+2s)}\frac{z^n}{n!}$$

$$= \frac{\Gamma_k(s)\Gamma_k(c-\gamma+s)}{\Gamma_k(\gamma)\Gamma_k(c-\gamma)}\times_{k,2}\Psi_2\left[\begin{array}{c}(c,k),(\gamma+s,k);\\ \\ (\beta,\alpha),(c+2s,k)\end{array},z\right]$$

□

Corollary 2.4. *Taking $s=k$ in Theorem 2.3, we have*

(19)

$$\int_0^{\infty} E_{k,\alpha,\beta}^{\gamma;c}(z;p)dp = \frac{\Gamma_k(c-\gamma+k)}{\Gamma_k(\gamma)\Gamma_k(c-\gamma)}{}_{k,2}\Psi_2\left[\begin{array}{c}(c,k),(\gamma+k,k);\\ \\ (\beta,\alpha),(c+2k,k)\end{array},z\right]$$

Corollary 2.5. *Taking the inverse Mellin transform on both sides of (13), we have the following result*

$$(20)\quad E_{k,\alpha,\beta}^{\gamma;c}(z;p) = \frac{1}{2\pi\iota\Gamma_k(\gamma)\Gamma_k(c-\gamma)}\int_{\upsilon-\iota\infty}^{\upsilon+\iota\infty}\Gamma_k(s)\Gamma_k(c-\gamma+s)$$

$$\times_{k,2}\Psi_2\left[\begin{array}{c}(c,k),(\gamma+s,k);\\ \\ (\beta,\alpha),(c+2s,k)\end{array},z\right]p^{-s}ds, \upsilon>0.$$

3. Some Derivative properties of extended k-Mittag-leffler function

In this section, we derive the extended Riemann-Liouville k-fractional differential formulas. Also, we prove some derivative properties of the extended k-Mittag-Leffler function.

Definition. 3.1 (see [14]) The extended Reimann-Liouville fractional derivative is given by

$$\mathfrak{D}_z^{\mu,p}\{f(z)\} = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}\exp[\frac{-pz^2}{t(z-t)}]dt, \tag{21}$$

where $\Re(\mu) < 0$, $\Re(p) > 0$ and for $m-1 < \Re(\mu) < m$ (where $m = 1, 2, 3, \cdots$), we have

$$\mathfrak{D}_z^{\mu,p}\{f(z)\} = \frac{d^m}{dz^m}\mathfrak{D}_z^{\mu,p}\{f(z)\}$$

$$= \frac{d^m}{dz^m}\left\{ \frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}\exp[\frac{-pz^2}{t(z-t)}]dt \right\}. \tag{22}$$

If $p = 0$, then we obtain the well-known Riemann-Liouville fractional derivative.

Definition. 3.2 For $k > 0$, we define the extended Riemann-Liouville k-fractional derivative as follows:

$${}_k\mathfrak{D}_z^{\mu,p}\{f(z)\} = \frac{1}{k\Gamma_k(-\mu)}\int_0^z f(t)(z-t)^{-\frac{\mu}{k}-1}\exp[\frac{-p^k z^2}{kt(z-t)}]dt, \tag{23}$$

where $\Re(\mu) < 0$, $\Re(p) > 0$ and for $m-1 < \Re(\mu) < m$ (where $m = 1, 2, 3, \cdots$), we have

$${}_k\mathfrak{D}_z^{\mu,p}\{f(z)\} = \frac{d^m}{dz^m}k^m.{}_k\mathfrak{D}_z^{\mu,p}\{f(z)\}$$

$$= \frac{d^m}{dz^m}\left\{ k^m\frac{1}{k\Gamma_k(-\mu+mk)}\int_0^z f(t)(z-t)^{\frac{-\mu+mk}{k}-1}\exp[\frac{-p^k z^2}{kt(z-t)}]dt \right\}. \tag{24}$$

If $p = 0$ in (23), then we obtain the well-known Riemann-Liouville k-fractional derivative formula (see [3]). Similarly, if $k = 1$ then we have 21.

Theorem 3.1. *Let $p \geq 0$, $\Re(mu) > \Re(\lambda) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $k > 0$, then*

$${}_k\mathfrak{D}_z^{\mu,p}\{z^{\frac{\lambda}{k}-1}E_{k,\alpha,\beta}^{c}(z)\} = \frac{z^{\frac{\mu}{k}-1}B_k(\lambda, c-\lambda)}{\Gamma_k(\mu-\lambda)}E_{k,\alpha,\beta}^{\lambda;c}(z;p). \tag{25}$$

Proof. Setting $\mu = \lambda - \mu$ in equation (23), we have

$${}_k\mathfrak{D}_z^{\mu,p}\{z^{\frac{\lambda}{k}-1}E_{k,\alpha,\beta}^{c}(z)\}$$

$$= \frac{1}{k\Gamma_k(\mu-\lambda k)}\int_0^z t^{\frac{\lambda}{k}-1}(z-t)^{\frac{-\lambda+\mu}{k}-1}E^c_{k,\alpha,\beta}(t)\exp[\frac{-p^k z^2}{kt(z-t)}]dt$$

$$= \frac{z^{\frac{-\lambda+\mu}{k}-1}}{k\Gamma_k(\mu-\lambda)}\int_0^z t^{\frac{\lambda}{k}-1}(1-\frac{t}{z})^{\frac{-\lambda+\mu}{k}-1}E^c_{k,\alpha,\beta}(t)\exp[\frac{-p^k z}{kt(1-\frac{t}{z})}]dt$$

Putting $u=\frac{t}{z}$, we obtain

$$ {}_k\mathfrak{D}^{\mu,p}_z\{z^{\frac{\lambda}{k}-1}E^c_{k,\alpha,\beta}(z)\} = \frac{z^{\frac{\mu}{k}-1}}{k\Gamma_k(\mu-\lambda)}\int_0^z u^{\frac{\lambda}{k}-1}(1-u)^{\frac{-\lambda+\mu}{k}-1}$$

$$\times E^c_{k,\alpha,\beta}(uz)\exp[\frac{-p^k}{ku(1-u)}]du. \tag{26}$$

Comparing the above result with (10), we get

$$ {}_k\mathfrak{D}^{\mu,p}_z\{z^{\frac{\lambda}{k}-1}E^c_{k,\alpha,\beta}(z)\} = \frac{z^{\frac{\mu}{k}-1}B_k(\lambda,c-\lambda)}{\Gamma_k(\mu-\lambda)}E^{\lambda;c}_{k,\alpha,\beta}(z;p). \tag{27}$$

Which completes the proof. □

Theorem 3.2. *For the extended k-Mittag-Leffler function, we have the following result:*

$$\frac{d^n}{dz^n}\{E^{\gamma;c}_{k,\alpha,\beta}(z;p)\} = (c)_{n,k}E^{\gamma+nk;c+nk}_{k,\alpha,\beta+n\alpha}(z;p), n\in\mathbb{N}. \tag{28}$$

Proof. Taking the derivative of the extended k-Mittag-Leffler function with respect to z, we have

$$\frac{d}{dz}\{E^{\gamma;c}_{k,\alpha,\beta}(z;p)\} = cE^{\gamma+k;c+k}_{k,\alpha,\beta+\alpha}(z;p). \tag{29}$$

Again taking derivative with respect to z, we get

$$\frac{d^2}{dz^2}\{E^{\gamma;c}_{k,\alpha,\beta}(z;p)\} = cE^{\gamma+2k;c+2k}_{k,\alpha,\beta+2\alpha}(z;p). \tag{30}$$

Continuing in this way up to n times, we get the required result. □

Theorem 3.3. *For the extended k-Mittag-Leffler function, we have the following differentiation formula satisfies:*

$$k^n\frac{d^n}{dz^n}\{z^{\frac{\beta}{k}-1}E^{\gamma;c}_{k,\alpha,\beta}(\omega z^{\frac{\alpha}{k}};p)\} = z^{\frac{\beta}{k}-n-1}E^{\gamma;c}_{k,\alpha,\beta-nk}(\omega z^{\frac{\alpha}{k}};p) \tag{31}$$

Proof. From the definition of the extended k-Mittag-Leffler function

$$\frac{d^n}{dz^n}\{z^{\frac{\beta}{k}-1}E^{\gamma;c}_{k,\alpha,\beta}(\omega z^{\frac{\alpha}{k}};p)\}$$

$$
\begin{aligned}
&= \frac{d^n}{dz^n}\sum_{n=0}^{\infty}\frac{B_k(\gamma+nk,c-\gamma)}{B_k(\gamma,c-\gamma)}\frac{(c)_{n,k}\lambda^n}{\Gamma_k(n\alpha+\beta)}\frac{z^{\frac{n\alpha+\beta}{k}-1}}{n!}\\
&= \sum_{n=0}^{\infty}\frac{B_k(\gamma+nk,c-\gamma)}{B_k(\gamma,c-\gamma)}\frac{(c)_{n,k}\lambda^n}{\Gamma_k(n\alpha+\beta)}\frac{z^{\frac{n\alpha+\beta}{k}-n-1}}{n!}\\
&\times \frac{(n\alpha+\beta-k)(n\alpha+\beta-2k)\cdots(n\alpha+\beta-nk)}{k^n}\\
&= z^{\frac{\beta}{k}-n-1}\sum_{n=0}^{\infty}\frac{B_k(\gamma+nk,c-\gamma)}{B_k(\gamma,c-\gamma)}\frac{(c)_{n,k}}{\Gamma_k(n\alpha+\beta)}\frac{\lambda^n z^{\frac{n\alpha}{k}}}{k^n n!}\frac{\Gamma_k(n\alpha+\beta)}{\Gamma_k(n\alpha+\beta-nk)}
\end{aligned}
$$

This implies that

$$
k^n\frac{d^n}{dz^n}\{z^{\frac{\beta}{k}-1}E^{\gamma;c}_{k,\alpha,\beta}(\omega z^{\frac{\alpha}{k}};p)\} = z^{\frac{\beta}{k}-n-1}E^{\gamma;c}_{k,\alpha,\beta-nk}(\omega z^{\frac{\alpha}{k}};p).
$$

Which is the required result. □

remark 3.4. *In this paper, we introduced the extended k-Mittag-Leffler function and derived its various properties. We conclude that if we letting $k = 1$ through out in this paper, then we have the earlier proved result of extended Mittag-Leffler function* [13]. *Similarly, if $p = 0$, then we get the result of k-Mittag-Leffler function.*

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